The Japanese Economic Review
Vol. 49, No. 2, June 1998

ON EQUILIBRIUM DYNAMICS UNDER EXTERNALITIES IN A MODEL OF ECONOMIC DEVELOPMENT*

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This paper studies a discrete-time version of a model of economic development proposed by Lucas, in which the average stock of human capital produces a positive externality in the production of the physical capital good. It establishes the existence of equilibrium programmes from arbitrary initial conditions, and of an equilibrium steady-state programme. The principal results are concerned with the *global* dynamic behaviour of equilibrium programmes off the steady state. They show that the asymptotic growth rates of consumption, physical and human capital on any equilibrium programme will equal the growth rates of the respective variables along the equilibrium steady state.
JEL Classification Numbers: C62, D90, O41.

1. Introduction

In recent years, the literature on "endogenous growth" has increased quite rapidly, and we now have a variety of models that attempt to explain the observable patterns of growth over time and across countries.

Most of these models use as the analytic framework some version of a two-sector model in which one sector produces a good that can be used for consumption or for investment in physical capital, and another sector produces new human capital or knowledge which constitutes the endogenous technical change that promotes growth in per capita incomes. The characterization of an optimal or equilibrium *steady state* in these models is fairly clear by this stage. However, it would not be unfair to say that the study of the dynamics of equilibrium or optimal programmes off the steady state is still at a preliminary stage.

It is worth mentioning that some of the leading contributors to this literature have recognized the importance of understanding the "transition dynamics" of these growth models. In his evaluation of the growth paths generated by the Solow growth model (interpreted as representative agent perfect-foresight equilibria), Lucas (1988, p. 11) comments: "What of economies that begin *off* the balanced path — surely the normal case? Cass showed — and this is exactly why the balanced path is interesting to us — that for *any* initial capital $K(0) > 0$, the optimal capital-consumption path $(K(t), c(t))$ will converge to the balanced path asymptotically. That is, the balanced path will be a good approximation to any actual path 'most' of the time." Romer (1990, p. S97), in

* A version of this paper was presented at the Conference on "Endogenous Growth and Nonlinear Economic Dynamics" in Meiji-Gakuin University, Tokyo, in November 1993. Comments received from participants at the Conference, especially Kazuo Mino, the discussant of the paper, are gratefully acknowledged. I would also like to thank John Hartwick, Debraj Ray and Manuel Santos for helpful conversations.

Published by Blackwell Publishers, 108 Cowley Road, Oxford OX4 1JF, UK.

examining the welfare properties of his model of endogenous technological change (in which growth is driven by the accumulation of knowledge in the disembodied form of new designs), comments: "Within the confines of the model, the social optimum can be achieved by subsidizing the accumulation of A. Demonstrating this result rigorously starting from arbitrary initial conditions forces the analysis to depart from consideration of balanced growth paths. Any intervention designed to move an economy from one balanced growth path to another must consider the transition dynamics along the way, and an explicit analysis of these dynamics is beyond the scope of this paper."

However, neither Lucas (1988) nor Romer (1990) provides an analysis of the transition dynamics in their respective models.[1)] The reason appears to be that the dynamics of the models proposed by them involve two state variables (physical capital and human capital or knowledge) in an *essential way*. That is, one cannot "reduce" the dynamic analysis to one that involves the law of motion of a single state variable. The analyses of such dynamical systems are inherently more difficult,[2)] as is clear from the literature on the theory of optimal intertemporal allocation with heterogeneous capital goods.[3)]

The purpose of this paper is to provide a systematic analysis of a model of economic development proposed by Lucas (1988), in which technical change is endogenously produced through the process of human capital accumulation. Households can allocate their available labour in each period between a physical good sector and a human capital sector. In the former sector, a consumption/physical capital good is produced by using inputs of capital and labour (in efficiency units), so that the accumulation of human capital in the latter sector provides a labour-augmenting technical progress in the production of the physical good (by increasing the efficiency units of labour). However, in addition to this, the *average* level of human capital (for the economy) provides a positive externality in the production of the former sector.

Diminishing returns are absent in the technology of human capital accumulation, which allows sustained growth of output per worker in this model. The presence of the externality in the production of the physical good implies that an *optimal* programme (obtained by solving a planning problem in which the externality is internalized) differs from an *equilibrium* programme (which is obtained in a decentralized setting, when beliefs of agents regarding the externality are fulfilled). The focus of this paper is on equilibrium programmes.

We establish the existence of equilibrium programmes from arbitrary initial conditions by applying a fixed-point argument in the space of all sequences (the set of beliefs being a subset of this space). After noting some preliminary properties of equilibrium programmes (relating to patterns of human capital accumulation and the Ramsey–Euler equations), we establish the existence of an equilibrium steady-state

1) "The dynamics of this system are not as well understood as those of the one-good model ..." (Lucas 1988, p. 25). "The treatment of behavior off balanced paths is largely conjecture ..." (Lucas 1988, p. 20). "By focusing only on balanced growth paths, the analysis neglects the transient dynamics that arise when the economy starts from a ratio of K to A that differs from the ratio that is maintained along the balanced growth path" (Romer 1990, p. S90).

2) This observation is in agreement with the following evaluation of Mulligan and Sala-i-Martin (1993, p. 740): "Due to its analytical difficulty, however, these transitional dynamics are always left unexplained."

3) See McKenzie (1986) for a comprehensive survey of this literature.

programme. The principal results of the paper are concerned with the dynamic behaviour of equilibrium programmes *off* the steady state. We show that the asymptotic growth rates of consumption, physical and human capital on any (interior) equilibrium programme will equal the growth rates of the respective variables along the equilibrium steady state.

We now relate our results to those available in the literature.[4)] The dynamic analysis of a model, which is the same as that studied by Lucas (1988) except that *the production externality is absent*, is by now well understood. Uzawa (1965), Caballé and Santos (1993) (in a continuous-time framework) and Mitra and Sihag (1993) (in a discrete-time framework) provide complete analyses of transition dynamics in different versions of this model. (Uzawa assumed that the utility function of the (representative) agent was linear; Caballé and Santos and Mitra and Sihag worked with strictly concave utility functions.[5)]) There are two aspects in which our present exercise can be distinguished from the above literature. First, in the absence of the externality, the notions of "equilibrium" and "optimum" coincide, and one can draw on the rich literature on optimal growth (the planning problem) to analyse the decentralized equilibrium solution (and, indeed, all three of the above contributions do precisely this). Second (at least for interior solutions), the dynamic analysis of the two state variables (physical and human capital) can be "reduced" to the analysis of a single variable (the ratio of labour (in efficiency units) devoted to physical good production to the physical capital stock).

The "transition dynamics" in models in which some production externality is present have not been investigated in the same way. Chamley (1993) and Benhabib and Farmer (1994) consider alternate versions of two-sector models with externalities that differ from those considered by Lucas. However, their off-the-steady-state dynamic analyses are confined to obtaining local results which hold in the neighbourhood of the steady state. Mulligan and Sala-i-Martin (1993) consider a general two-sector model with externalities which includes the model of Lucas as a special case. However, their study of the transition dynamics is confined to models without externalities.[6)] Furthermore, their paper provides only numerical simulations of the dynamics of such of a model: it does not offer analytical results.[7)]

4) A question in this literature that we do not address is the non-uniqueness, and more generally the "indeterminacy", of equilibrium programmes in models with dynamic externalities. This important issue has been the focus of a number of papers in recent years; see especially Benhabib and Farmer (1994), Boldrin and Rustichini (1994) and Benhabib and Perli (1994). Specifically, Xie (1994) uses special pairs of the utility and production functions to examine the indeterminacy problem in the Lucas model.

5) Mino (1996) and Bond *et al*. (1996) examine a more general model, but neither paper offers a complete global dynamic analysis.

6) It is, perhaps, worth noting that (although they exclude externalities in their simulation studies) Mulligan and Sala-i-Martin (1993) consider the case where, in addition to human capital and labour, physical capital may also be an input in the production of human capital, a situation not allowed by Lucas (1988). However, this case (without externalities) is given a definitive analytical treatment in Caballé and Santos (1993).

7) Mulligan and Sala-i-Martin (1993, p. 759) note in their "Interesting Result 6" that "The Uzawa (1965)–Lucas (1988) model of endogenous growth is globally saddle-path stable." This is however, immediately followed by footnote 24: "This result comes from extensive experimentation with all kinds of parameters and is not based on a formal proof. Strictly speaking, interesting result 6 should say, 'We have not been able to find parameters for which the Uzawa–Lucas model was not globally saddle-path stable.'"

These contributions use continuous-time formulations in contrast to our discrete-time version. The literature that has developed over the past fifteen years in the theory of optimal intertemporal allocation (relying on the mathematical contributions on "chaotic dynamics") has taught us that discrete-time models can behave quite differently from their continuous-time counterparts. Thus, in developing our results, we have not relied on the continuous-time literature, whatever there is of it. The analytical techinques that we have found most useful for our study are those developed by Brock and Gale (1969) to examine the nature of optimal growth paths in a model of *exogenous* technical change. We feel that the methods used in our paper (based on those of Brock and Gale) might be useful in studying the transition dynamics in other two-sector models of endogenous technical change.

2. The Model

Consider an infinitely lived representative agent with initial stocks of physical capital (K) and human capital (A), and one unit of labour in each period. The labour is allocated between the physical goods production sector (h) and the human capital production sector ($1 - h$).

Production of physical goods is determined by a production function $F : \mathfrak{R}^3_+ \to \mathfrak{R}_+$, the arguments in the production function being the amounts of physical capital (K), labour in efficiency units (L) and the average (social) stock of human capital (E). For the representative agent, this last input is a production externality. Human capital production is determined by a linear production function with $v > 0$, the constant (average and marginal) productivity of labour in "efficiency" units. We will assume that there is no depreciation of either type of capital.

In this model, human capital creates labour-augmenting technical progress in the production of physical goods. This technical change is endogenously produced by allocating labour to the production of human capital.

Consumption of the good produced in the physical goods sector yields felicity to the agent as determined by a welfare function, $w : \mathfrak{R}_+ \to \mathfrak{R}_+$. The agent discounts future welfare by the discount factor $0 < \delta < 1$ and is interested in obtaining the maximum discounted sum of welfares from his given initial stocks, subject to feasibility constraints.

In formulating his maximization problem, the representative agent does not know the magnitude of the production externality; that is, the average stock of human capital in the economy at each date. Instead, he has "beliefs" about this magnitude, and he formulates his optimization problem based on these beliefs. Thereby an actual pattern of average human capital accumulation is generated for the economy. An equilibrium is a situation where the beliefs turn out to be correct.

We now proceed to develop the notion of an equilibrium more formally.

A *belief* is any sequence $\{E_t\}_0^\infty$ satisfying $E_t > 0$ for $t \geqslant 0$. We interpret E_t to be a representative agent's belief of what the average stock of human capital for the economy will be at time t.

Given any initial stocks of physical and human capital $(\mathbf{K}, \mathbf{A}) \geqslant 0$, and a belief $\{E_t\}_0^\infty$, a *plan* from $(\mathbf{K}, \mathbf{A})$ is a sequence $\{K_t, A_t, h_{t+1}\}_0^\infty$ satisfying

$$(K_0, A_0) = (\mathbf{K}, \mathbf{A}); \; 0 \leqslant h_{t+1} \leqslant 1 \qquad \text{for } t \geqslant 0$$

$$0 \leqslant K_{t+1} \leqslant F(K_t, A_t h_{t+1}, E_t) + K_t \qquad \text{for } t \geqslant 0$$

$$A_{t+1} = A_t + \upsilon A_t(1 - h_{t+1}) \qquad \text{for } t \geqslant 0.$$

We associate with such a plan a *consumption sequence* $\{C_{t+1}\}_0^\infty$, defined by

$$C_{t+1} = F(K_t, A_t h_{t+1}, E_t) + K_t - K_{t+1} \qquad \text{for } t \geqslant 0.$$

Given $(\mathbf{K}, \mathbf{A}) \geqslant 0$, and a belief $\{\hat{E}_t\}_0^\infty$, an *optimal plan* from $(\mathbf{K}, \mathbf{A})$ is a plan $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ (with the belief $\{\hat{E}_t\}_0^\infty$) such that, for every plan $\{K_t, A_t, h_{t+1}\}_0^\infty$ from $(\mathbf{K}, \mathbf{A})$, with the same belief $\{\hat{E}_t\}_0^\infty$,

$$\sum_0^\infty \delta^t w(F(K_t, A_t h_{t+1}, \hat{E}_t) + K_t - K_{t+1}) \leqslant \sum_0^\infty \delta^t w(F(\hat{K}_t, \hat{A}_t \hat{h}_{t+1}, \hat{E}_t) + \hat{K}_t - \hat{K}_{t+1}).$$

An *equilibrium programme* from $(\mathbf{K}, \mathbf{A}) \geqslant 0$ is a sequence $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}, \hat{E}_t\}_0^\infty$ such that

(i) $\{\hat{E}_t\}_0^\infty$ is a belief;
(ii) $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ is a plan from $(\mathbf{K}, \mathbf{A})$, given the belief $\{\hat{E}_t\}_0^\infty$;
(iii) if $\{K_t, A_t, h_{t+1}\}_0^\infty$ is any plan from $(\mathbf{K}, \mathbf{A})$, given the belief $\{\hat{E}_t\}_0^\infty$, then

$$\sum_0^\infty \delta^t w(F(K_t, A_t, h_{t+1}, \hat{E}_t) + K_t - K_{t+1})$$

$$\leqslant \sum_0^\infty \delta^t w(F(\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}, \hat{E}_t) + \hat{K}_t - \hat{K}_{t+1});$$

(iv) $\hat{E}_t = \hat{A}_t$ for $t \geqslant 0$.

In words, an equilibrium programme is a belief and a plan (given the belief) such that (a) it is an optimal plan given the belief, and (b) the belief is correct.

Thus, if $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}, \hat{E}_t\}_0^\infty$ is an equilibrium programme from $(\mathbf{K}, \mathbf{A}) \geqslant 0$, then $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ solves the following maximization problem:

$$\text{Max} \sum_0^\infty \delta^t w(F(K_t, A_t h_{t+1}, \hat{A}_t) + K_t - K_{t+1})$$

$$\text{subject to } (K_0, A_0) = (\mathbf{K}, \mathbf{A}), \qquad 0 \leqslant h_{t+1} \leqslant 1, \qquad \text{for } t \geqslant 0$$

$$0 \leqslant K_{t+1} \leqslant F(K_t, A_t h_{t+1}, \hat{A}_t) + K_t - K_{t+1} \qquad \text{for } t \geqslant 0$$

$$A_{t+1} = A_t + \upsilon A_t(1 - h_{t+1}) \qquad \text{for } t \geqslant 0.$$

That is, if $\{\hat{A}_t\}_0^\infty$ itself is the belief, then $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ is an optimal plan among all plans with that belief (from the same initial stocks). Note that, if $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}, \hat{E}_t\}_0^\infty$ is an equilibrium programme from $(\mathbf{K}, \mathbf{A})$, then the equilibrium programme can, in fact, be described by the sequence $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$. We shall use this convention in the rest of the paper.

The following assumptions on F and w are maintained throughout the paper:

(A) $F(K, L, E) = K^\beta L^{1-\beta} E^\gamma$ for $(K, L, E) \in \mathfrak{R}_+^3$, where $0 < \beta < 1$ and $\gamma > 0$
(B) $w(C) = C^{1-\sigma}/(1 - \sigma)$ for $C \in \mathfrak{R}_+$, where $0 < \sigma < 1$.

The Cobb–Douglas production function and the isoelastic welfare function have figured prominently in the literature on growth with endogenous technical change (see e.g. Lucas 1988; Romer 1990; Rebelo 1991). More general forms of welfare and production functions are allowed in Caballé and Santos (1993) and in Mitra and Sihag (1993). Uzawa (1965) also allows a more general form of the production function but uses a linear welfare function.

Remark. Notice that in Assumption (B) we restrict σ to (0, 1), unlike much of the literature in this area, which also allows for $\sigma > 1$. For $\sigma > 1$ (as well as for $w(C) = \ln C$, the limiting case of (B) as $\sigma \to 1$), the welfare function is not defined at zero. A *rigorous* treatment of this case leads to considerable additional analytical work in establishing the existence results, starting with the result on existence of an optimal plan given a belief (Proposition 1). While such a treatment is possible (following, for instance, the approach of Ekeland and Scheinkman 1986), we felt justified in restricting our attention to the case in which σ is in (0, 1) for the gain in analytical simplicity, and because the focus of this paper is the nature of transition dynamics, and not the existence of optimal or equilibrium plans.

3. The Existence of an Equilibrium

It can be shown that there exists an equilibrium from every specification of initial stocks $(\mathbf{K}, \mathbf{A}) \gg 0$. This is established by using the Tychonoff fixed-point theorem in the space of all sequences. We indicate below the main steps in this argument.

An elementary observation, following from the definition of an equilibrium, is that, if $\{K_t, A_t, h_{t+1}\}_0^\infty$ is an equilibrium programme from $(\mathbf{K}, \mathbf{A})$, then

$$A_{t-1} \leqslant A_t \leqslant A_{t-1}(1+\upsilon), \qquad A_0 = \mathbf{A}.$$

Thus, in proving the existence of an equilibrium programme, the beliefs $\{E_t\}_0^\infty$ that we should be concerned with are those that satisfy

$$E_{t-1} \leqslant E_t \leqslant E_{t-1}(1+\upsilon), \qquad E_0 = \mathbf{A}. \tag{1}$$

Given this restriction on beliefs, a convenient boundedness property of plans (given the beliefs) can be established.

To this end, define $\mu = (1-\beta+\gamma)/(1-\beta)$, $\theta = (1+\upsilon)^\mu$. (Note that $\mu > 1$, $\theta > (1+\upsilon)$.)

Lemma 1: *Given any* $(\mathbf{K}, \mathbf{A}) \geqslant 0$ *and any belief* $\{E_t\}_0^\infty$ *satisfying* (1), *there is* $B > 0$ *such that, if* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is a plan from* $(\mathbf{K}, \mathbf{A})$, *then*

$$K_t,\ C_t \leqslant B\theta^t \qquad \textit{for } t \geqslant 1.$$

The next step is to obtain an optimal plan from every initial stock configuration $(\mathbf{K}, \mathbf{A})$, and belief $\{E_t\}_0^\infty$ satisfying (1). For this, we need a familiar condition involving the discount factor, δ, the maximum growth factor, θ, and the elasticity of the welfare function, $(1-\sigma)$.

Proposition 1: *Given any* $(\mathbf{K}, \mathbf{A}) \geqslant 0$ *and any belief* $\{E_t\}_0^\infty$ *satisfying* (1), *there is an optimal plan from* $(\mathbf{K}, \mathbf{A})$, *given the belief* $\{E_t\}_0^\infty$, *if*

(C) $\delta\theta^{1-\sigma} < 1$.

Furthermore, an optimal plan is unique when $(\mathbf{K}, \mathbf{A}) \gg 0$.

For the rest of our analysis, we maintain (C) as an assumption.

The final step is to find a belief that is correct (self-fulfilling). Let Ω be the space of sequences $\{\omega_t\}_0^\infty$ with $\omega_t \in \Re$ for $t \geqslant 0$. We can find a locally convex topology for Ω which is equivalent to the topology of co-ordinate wise convergence. Let $(\mathbf{K}, \mathbf{A}) \gg 0$ be given and let F be the set of sequences $\{E_t\}_0^\infty$ satisfying (1). Then F is a non-empty, convex and compact subset of Ω. Given any element of F, we obtain a unique optimal plan $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ by Proposition 1. One can show that optimal plans depend continuously on beliefs. Using a projection map, we obtain the sequence $\{\hat{A}_t\}_0^\infty$ associated with the sequence $\{E_t\}_0^\infty$ in F. Since the projection map is continuous, the mapping of $\{E_t\}_0^\infty$ to $\{\hat{A}_t\}_0^\infty$ is also continuous. Further, by feasibility of $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ from $(\mathbf{K}, \mathbf{A})$, $\{\hat{A}_t\}_0^\infty$ is also in F. Thus, applying the Tychonoff fixed-point theorem, there is $\{E_t\}_0^\infty$ in F such that the optimal plan $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ from $(\mathbf{K}, \mathbf{A})$ satisfies $\hat{A}_t = E_t$ for $t \geqslant 0$. It is straightforward to verify from this that $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ is an equilibrium programme from $(\mathbf{K}, \mathbf{A}) \gg 0$.

Theorem 1: *Given any* $(\mathbf{K}, \mathbf{A}) \gg 0$, *there exists an equilibrium programme* $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}^\infty$.

4. Basic Properties of Equilibrium Programmes

4.1 Accumulation of physical capital

If $\{K_t, A_t, h_{t+1}\}_0^\infty$ is an equilibrium programme from $(\mathbf{K}, \mathbf{A}) \gg 0$, then clearly $C_t > 0$ for $t \geqslant 1$ (since $w'(C) \to \infty$ as $C \to 0$). This implies that $K_t > 0$ for $t \geqslant 0$. Also, since $A_{t+1} \geqslant A_t$ and $A_0 > 0$, we have $A_t > 0$ for $t \geqslant 0$.

The allocation of labour to the physical good sector (h_{t+1}) must clearly be positive for a subsequence of periods. Further, if it were not positive for all periods, we could find a period τ such that $h_{\tau+1} = 0$ and $h_{\tau+2} > 0$. However, now we can produce a small perturbation in periods $\tau+1$, $\tau+2$ (increasing the allocation of labour to the physical good sector in period $\tau+1$ and reducing it in period $\tau+2$, leaving other periods unchanged) which will be better for the agent than the original plan, given the belief $\{A_t\}_0^\infty$. Thus, we can conclude that $h_{t+1} > 0$ for $t \geqslant 0$.

On the basis of the above information, we can obtain a Ramsey–Euler equation relating the intertemporal marginal rate of substitution to the marginal product of capital in the physical good sector. This represents the standard trade-off of more consumption today versus an accumulation of capital today and more consumption tomorrow. All of the above discussion can be summarized in the following result.

Proposition 2: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then*

(*i*) $(C_{t+1}, K_t, A_t, h_{t+1}) \gg 0$ *for* $t \geqslant 0$;

(*ii*) $w'(C_t)/\delta w'(C_{t+1}) = [1 + D_1F(K_t, A_t h_{t+1}, A_t)]$ *for* $t \geqslant 1$.

4.2 Accumulation of human capital

The pattern of human capital accumulation is somewhat harder to characterize. Assuming a "δ-productivity" condition in the human capital sector, we can show that, in equilibrium, human capital must be accumulated at least for a subsequence of periods.

We state this δ-productivity condition as follows:

$$\delta(1+v) > 1$$

and we maintain it in the rest of our analysis.

Proposition 3: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then there is a subsequence* $\{t_s\}_{s=1}^\infty$ *of periods such that, if* $\tau = t_s$, *then*

$$A_{\tau+1} > A_\tau.$$

It appears that there is no reason to rule out $h_{t+1} = 1$ for *some* period t ($A_{t+1} = A_t$), a period of no human capital accumulation in equilibrium. We will, however, focus on equilibria in which this phenomenon is ruled out.[8] An equilibrium programme will be called *interior* if

$$h_{t+1} < 1 \qquad \text{for } t \geqslant 0. \tag{2}$$

For interior equilibria, one can derive another Ramsey–Euler equation arising from the trade-off of more consumption today (by less human capital accumulation today) versus more consumption tomorrow (by less human capital accumulation tomorrow).

Proposition 4: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an interior equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then*

$$w'(C_t)/\delta w'(C_{t+1}) = (1+v)[D_2F(K_t, A_t h_{t+1}, A_t)/D_2F(K_{t-1}, A_{t-1}h_t, A_{t-1})] \qquad \textit{for } t \geqslant 1.$$

4.3 The Ramsey–Euler equations

We can combine the Ramsey–Euler equations obtained in Propositions 2 and 4 in a more convenient form which will help us in our analysis in the following sections. For this purpose, define

$$x_t = A_t^\mu \qquad \text{for } t \geqslant 0$$

Proposition 5: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an interior equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then*

(*i*) $(1/\delta)(C_{t+1}/C_t)^\sigma = [1 + \beta(h_{t+1}x_t/K_t)^{1-\beta}] \qquad \textit{for } t \geqslant 1$

(*ii*) $(1/\delta)(C_{t+1}/C_t)^\sigma = (1+v)\dfrac{(h_t x_{t-1}/K_{t-1})^\beta}{(h_{t+1}x_t/K_t)^\beta}(A_t/A_{t-1})^{\mu-1} \qquad \textit{for } t \geqslant 1$

8) In a framework without externalities, Mitra and Sihag (1993) show how the more general case (allowing for $h_{t+1} = 1$ for some periods) can be handled. Our assessment is that those methods should be applicable in the present context as well.

5. An Equilibrium Steady State

In this section we establish the existence of an equilibrium programme which also happens to be a steady state, in the sense that consumption, physical capital and human capital grow at constant rates over time.

Formally, an equilibrium programme $\{K_t, A_t, h_{t+1}\}$ from $(\mathbf{K}, \mathbf{A}) \geqslant 0$ is an *equilibrium steady state programme* from $(\mathbf{K}, \mathbf{A})$ if there are numbers g_C, g_K, g_A such that

$$K_{t+1} = g_K K_t \qquad \text{for } t \geqslant 0,$$

$$A_{t+1} = g_A A_t \qquad \text{for } t \geqslant 0,$$

$$C_{t+1} = g_C C_t \qquad \text{for } t \geqslant 1.$$

In establishing our existence result, it helps to know the growth rates of physical and human capital and the composition of these capital stocks that would prevail in an equilibrium steady state.

Note that, for an equilibrium steady-state programme from $(\mathbf{K}, \mathbf{A}) \gg 0$, we must have $g_A < (1 + v)$ by Proposition 2 and $g_A > 1$ by Proposition 3. Thus, an equilibrium steady-state programme $\{K_t, A_t, h_{t+1}\}_0^\infty$ from $(\mathbf{K}, \mathbf{A}) \gg 0$ is necessarily interior, with $h_{t+1} = 1 - [(g_A - 1)/v]$ for $t \geqslant 0$. Using this in the first Ramsey–Euler equation of Proposition 5, we can infer that (x_t/K_t) must be constant over time and so $g_A^\mu = g_K$, which implies $g_K > g_A$.

Using the basic feasibility equation $C_{t+1} = K_t^\beta (h_{t+1}A_t)^{1-\beta} A_t^\gamma + K_t - K_{t+1}$, we obtain

$$(C_{t+1}/K_t) + (g_K - 1) = h_{t+1}^\beta (x_t/K_t)^{1-\beta}.$$

Thus, (C_{t+1}/K_t) is also a constant over time, so that $g_C = g_K$. We call this common growth rate G, and denote g_A by g. Then $G = g^\mu$, as we have noted before. Also, using the second Ramsey–Euler equation of Proposition 5, we can infer that

$$g_C^\sigma = g_A^{\mu-1}\delta(1 + v).$$

Thus, combining the two pieces of information, we get

$$g^{\mu\sigma} = G^\sigma = g^{\mu-1}\delta(1 + v).$$

Denoting $[1 - \mu(1 - \sigma)]$ by ρ, we have $0 < \rho < 1$ by assumption (C), and

$$g = [\delta(1 + v)]^{1/\rho}; \qquad G = [\delta(1 + v)]^{\mu/\rho}. \tag{3}$$

The above discussion can be summarized in the following result.

Proposition 6: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an equilibrium steady-state programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then, defining g and G as in* (3), *we have*

$$g_A = g, \qquad g_K = g_C = G;$$

$$h_{t+1} = 1 - [(g - 1)/v] \qquad \textit{for } t \geqslant 0;$$

$$(A_t^\mu/K_t) = \frac{\{[(1/\delta)G^\sigma - 1]/\beta\}^{1/(1-\beta)}}{\{1 - [(g - 1)/v]\}} \qquad \textit{for } t \geqslant 0.$$

Armed with this result, we can proceed to state the existence result of this section as follows.

Theorem 2: *If* $(\mathbf{K}, \mathbf{A}) \gg 0$, *g and G are defined as in* (3) *and*

$$(\mathbf{A}^{\mu}/\mathbf{K}) = \frac{\{[(1/\delta)G^{\sigma} - 1]/\beta\}^{1/(1-\beta)}}{\{1 - [(g-1)/v]\}},$$

then the sequence $\{K_t, A_t, h_{t+1}\}_0^{\infty}$ *defined by*

$(K_0, A_0) = (\mathbf{K}, \mathbf{A})$;

$h_{t+1} = 1 - [(g-1)/v]$ *for* $t \geqslant 0$;

$A_{t+1} = A_t + vA_t(1 - h_{t+1})$ *for* $t \geqslant 0$;

$K_{t+1} = GK_t$ *for* $t \geqslant 0$.

is an equilibrium steady-state programme from $(\mathbf{K}, \mathbf{A})$.

6. Equilibrium Dynamics off the Steady State

In examining the dynamics of equilibrium programmes, we focus on the behaviour of the variable

$$z_t = (h_{t+1}A_t^{\mu}/K_t) \qquad \text{for } t \geqslant 0. \tag{4}$$

This is precisely the labour (in efficiency units)-to-capital ratio in the absence of any externality (that is, with $\gamma = 0$ (so that $\mu = 1$), as is the case in the models examined by Uzawa 1965, Caballé and Santos 1993 and Mitra and Sihag 1993). With the externality present, $\mu > 1$, and so z_t may be considered to be an "augmented" efficiency labour-to-capital ratio, the augmentation reflecting the effect on production of the physical good of the average (social) stock of human capital, which is taken as given by the representative agent in his decision-making.

If we concentrate on interior equilibrium programmes, then combining the two Ramsey–Euler equations of Proposition 5 yields the basic difference equation

$$z_t^{\beta} + \beta z_t = z_{t-1}^{\beta}(A_t/A_{t-1})^{\mu-1}(1+v). \tag{5}$$

This would be an entirely straightforward difference equation to deal with (in terms of describing the qualitative properties of z_t over time) were it not for the externality factor, which makes its presence felt (apart from the definition of the z_t factor itself) in the term $(A_t/A_{t-1})^{\mu-1}$. However, noting that $1 \leqslant (A_t/A_{t-1}) \leqslant (1+v)$ for all $t \geqslant 1$, we can still say quite a bit about the behaviour of z_t over time, by following the methods used to study one-dimensional stochastic (Markov) processes. That is, although (A_t/A_{t-1}) is actually determined simultaneously with z_t and is a non-stochastic variable, we can say something about the behaviour of z_t from (5) without knowing anything specific about the *actual* value of (A_t/A_{t-1}), except that it lies in the interval $[1, (1+v)]$.

It can be shown that there is an interval $I \equiv [\underline{z}, \bar{z}]$, with $0 < \underline{z} < \bar{z} < \infty$, which represents the "absorbing states" of the system, in the sense that, if $z_t \in$ I for some t, then $z_t \in I$ thereafter. Furthermore, if $z_0 > 0$ is not in I initially, then there is some finite date, t, at which $z_t \in I$. Thus, the complement of I (in $\Re_{++}$) represents the "transitory states" of the system, and eventually the dynamics of the variable z_t are entirely confined to the interval I. This discussion is summarized in the following result (see also Figure 1).

Proposition 7: *There exist numbers* $0 < \underline{z} < \overline{z} < \infty$, *such that, if* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an interior equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *and* $I = [\underline{z}, \overline{z}]$, *then*

(*i*) $z_t \in I$ *implies* $z_{t+1} \in I$ *for* $t \geqslant 0$;
(*ii*) $z_t \in I$ *for some* $t \geqslant 0$.

Exploiting this result, we can next examine the behaviour of the variable $y_t = (x_t/K_t) = (A_t^\mu/K_t)$. Using Proposition 7, and $h_{t+1} \leqslant 1$ for $t \geqslant 0$, we clearly have the sequence $\{y_t\}_0^\infty$ bounded below by a positive number. It is somewhat harder to establish that the sequence is bounded above. The key idea here is that, if y_t became unbounded for a subsequence of periods, then by Proposition 7, h_{t+1} goes to zero for that subsequence, and the agent can improve his discounted sum of welfares by switching to a regime (sufficiently far along the given subsequence) in which schooling is stopped altogether (for the rest of the future).

In order to establish this, a preliminary result is useful. This simply states that, for an interior equilibrium, the propensity to consume cannot approach unity.

Lemma 2: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an interior equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then*

$$\limsup_{t\to\infty}(C_t/Y_t) < 1$$

when $Y_t = F(K_{t-1}, A_{t-1}h_t, A_{t-1}) + K_{t-1}$ *for* $t \geqslant 1$.

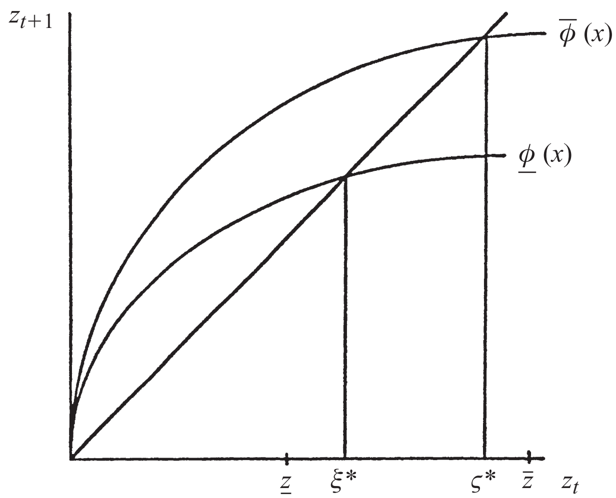

FIGURE 1.

Proposition 8: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an interior equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then*

$$0 < \liminf_{t\to\infty} y_t \leqslant \limsup_{t\to\infty} y_t < \infty.$$

Turning our attention next to the variable $v_t = (C_{t+1}/K_t)$, we can again use Proposition 7 to conclude that the sequence $\{v_t\}_0^\infty$ is bounded above. It is more difficult to establish that the sequence is bounded below by a positive number. The idea we exploit here is that, if v_t is close to zero for some t, then in fact the first Ramsey–Euler equation (and feasibility conditions) force the sequence $\{v_t\}_0^\infty$ to converge to zero at a geometric rate. However, this phenomenon can be shown to be "inefficient" and hence non-optimal for the representative agent.

The following lemma is useful in proving this result.

Lemma 3: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an interior equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *and* $\liminf_{t\to\infty} (C_t/K_t) = 0$, *then*

$$\sum_{t=1}^{\infty} (C_t/K_t) < \infty.$$

Proposition 9: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an interior equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then*

$$0 < \liminf_{t\to\infty} v_t \leqslant \limsup_{t\to\infty} v_t < \infty.$$

Combining the results of the previous two propositions, we can establish that asymptotic growth rates of A_t, K_t, C_t are well defined and equal to the growth rates of A_t, K_t, C_t (respectively) of the equilibrium steady-state programme of Section 5.[9]

Theorem 3: *If* $\{K_t, A_t, h_{t+1}\}_0^\infty$ *is an interior equilibrium programme from* $(\mathbf{K}, \mathbf{A}) \gg 0$, *then*

(*i*) $\lim_{t\to\infty} (A_t)^{1/t} = g$

(*ii*) $\lim_{t\to\infty} (K_t)^{1/t} = \lim_{t\to\infty} (C_t)^{1/t} = G$,

where g and G are defined by (3).

Appendix: Proofs

Proof of Lemma 1

Given $(\mathbf{K}, \mathbf{A}) \geqslant 0$ and the belief $\{E_t\}_0^\infty$ satisfying (1), we define

$$\hat{k} = [\theta''/(1-\theta')]^{1/(1-\beta)},$$

9) What we establish is weaker than the conjecture of Lucas (1988). If one could show that (A_t/g^t) and (K_t/G^t) converge to positive numbers as $t \to \infty$, then his conjecture would be established. Whether this stronger form of convergence holds remains an open question.

where $\theta' = (1/\theta)$, $\theta'' = \theta' \mathbf{A}^{(1-\beta+\gamma)}$ and

$$B = \max(\mathbf{K}, \hat{k}).$$

Now, for any plan $\{K_t, A_t, h_{t+1}\}_0^\infty$ from $(\mathbf{K}, \mathbf{A})$, given the belief $\{E_t\}_0^\infty$, we have

$$K_{t+1} \leqslant E_t^\gamma K_t^\beta A_t^{1-\beta} h_{t+1}^{1-\beta} + K_t,$$

so that, if we divide through by θ^{t+1}, and denote (K_t/θ^t) by k_t for $t \geqslant 0$,

$$k_{t+1} \leqslant \theta'' k_t^\beta + \theta' k_t.$$

It now readily follows that

$$k_t \leqslant B \text{ for all } t \geqslant 0,$$

so that $K_t \leqslant B\theta^t$ for $t \geqslant 0$, and $C_t \leqslant B\theta^t$ for $t \geqslant 1$. ■

Proof of Proposition 1

Given condition (C), if we define

$$b_{t+1} = B\theta^{t+1} \qquad \text{for } t \geqslant 0$$

(where B is given by Lemma 1), we have

$$w(b_{t+1}) = B'(\theta^{1-\sigma})^{t+1} \qquad \text{for } t \geqslant 0,$$

where $B' = B^{1-\sigma}/(1-\sigma)$. By Lemma 1, we also have, for every plan $\{K_t, A_t, h_{t+1}\}_0^\infty$ from $(\mathbf{K}, \mathbf{A}) \geqslant 0$ (given a belief $\{E_t\}_0^\infty$ satisfying (1)),

$$w(C_{t+1}) \leqslant w(b_{t+1}) \qquad \text{for } t \geqslant 0.$$

Thus, we have

$$\delta^t[w(b_{t+1}) - w(C_{t+1})] \geqslant 0 \qquad \text{for } t \geqslant 0$$

and

$$\sum_0^\infty \delta^t[w(b_{t+1}) - w(C_{t+1})] < \infty.$$

Then, by Brock and Gale (1969, lemma 2, p. 236), there is an optimal plan from $(\mathbf{K}, \mathbf{A})$, given the belief $\{E_t\}_0^\infty$. Uniqueness follows from the facts that w is strictly concave in C, F is concave in K and L (given E) and strictly concave in L (given K and E). ■

Proof of Theorem 1

Let $\Omega = \{\{\omega_t\}_0^\infty : \omega_t \in \Re \text{ for } t \geqslant 0\}$. For $\omega \in \Omega$, we can define $e^n(\omega) = \max(|\omega_0|, |\omega_1|, \ldots, |\omega_n|)$ for $n \geqslant 0$. For each n, e^n is a semi-norm on Ω. The formula

$$d(\omega, \tilde{\omega}) = \sum_{n=0}^\infty 2^{-n}\{e^n(\tilde{\omega} - \omega)/[1 + e^n(\tilde{\omega} - \omega)]\}$$

defines a (complete) metric on Ω. Thus, Ω is a locally convex (complete) metric space.

(Note that convergence in terms of the metric d is equivalent to co-ordinate wise convergence.)

Let $(\mathbf{K}, \mathbf{A}) \gg 0$ be given. Let F be the set of sequences $\{E_t\}_0^\infty$ satisfying (1). Then, it can be checked that F is a non-empty, convex and compact subset of Ω. (Checking the compactness of F is equivalent to checking the sequential compactness, since Ω is a metric space; and sequential compactness follows by using the standard Cantor diagonal argument.)

Given any element $\{E_t\}_0^\infty$ in F, we obtain a unique optimal plan $\{K_t, A_t, h_{t+1}\}_0^\infty$ by applying Proposition 1. Let ψ: $F \to \Omega^3$ be the map that associates the optimal plan $\{K_t, A_t, h_{t+1}\}_0^\infty$ which corresponds to the element $\{E_t\}_0^\infty$ in F. We want to show that ψ is a continuous map. (Note that for elements $(\omega^1, \omega^2, \omega^3)$ and $(\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3)$ in Ω^3, we can define the metric $\mathbf{d}((\omega^1, \omega^2, \omega^3), (\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3)) = \max(d(\omega^1, \tilde{\omega}^1), d(\omega^2, \tilde{\omega}^2), d(\omega^3, \tilde{\omega}^3))$.)

Let $\{E_t^n\}_0^\infty$ be a sequence of elements in F $(n = 1, 2, 3, \ldots)$ converging (in terms of the metric $\mathbf{d}$) to the element $\{E_t\}_0^\infty$. Let $\{K_t^n, A_t^n, h_{t+1}^n\}_0^\infty$ be the sequence of elements in Ω^3 $(n = 1, 2, 3, \ldots)$ defined by the map ψ. Finally, let $\{K_t, A_t, h_{t+1}\}_0^\infty = \psi(\{E_t\}_0^\infty)$. We want to show that $\mathbf{d}(\{K_t^n, A_t^n, h_{t+1}^n\}_0^\infty, \{K_t, A_t, h_{t+1}\}_0^\infty) \to 0$ as $n \to \infty$. If this did *not* hold, then we could find $\bar{\varepsilon} > 0$ and a subsequence n' (of n) such that $\mathbf{d}(\{K_t^{n'}, A_t^{n'}, h_{t+1}^{n'}\}_0^\infty, \{K_t, A_t, h_{t+1}\}_0^\infty) \geqslant \bar{\varepsilon}$ for all n'. By using Lemma 1 and the Cantor diagonal process, we can obtain a convergent subsequence of n' (retain notation) converging to $\{\overline{K}_t, \overline{A}_t, \overline{h}_{t+1}\}_0^\infty$. Continuity of F ensures that $\{\overline{K}_t, \overline{A}_t, \overline{h}_{t+1}\}_0^\infty$ is a plan from $(\mathbf{K}, \mathbf{A})$ given the belief $\{E_t\}_0^\infty$. Continuity of $\mathbf{d}$ ensures that $\mathbf{d}(\{K_t, A_t, h_{t+1}\}_0^\infty, \{\overline{K}_t, \overline{A}_t, \overline{h}_{t+1}\}_0^\infty) \geqslant \bar{\varepsilon}$. Therefore, the uniqueness of an optimal plan from $(\mathbf{K}, \mathbf{A})$, given the belief $\{E_t\}_0^\infty$, implies the existence of $\varepsilon > 0$, such that

$$\sum_0^\infty \delta^t w(C_{t+1}) \geqslant \sum_0^\infty \delta^t w(\overline{C}_{t+1}) + \varepsilon. \tag{+}$$

Using Lemma 1 and condition (C), one can find T such that, for every belief $\{E'_t\}_0^\infty$ satisfying (1), and every plan $\{K'_t, A'_t, h'_{t+1}\}_0^\infty$ from $(\mathbf{K}, \mathbf{A})$, given the belief, we have

$$\sum_T^\infty \delta^t w(C'_{t+1}) < (\varepsilon/5).$$

Then, we can choose N_1 large enough so that for $n' \geqslant N_1$ we have

$$\sum_0^T \delta^t w(\overline{C}_{t+1}) \geqslant \sum_0^T \delta^t w(C_{t+1}^{n'}) - (\varepsilon/5).$$

This implies, for $n' \geqslant N_1$,

$$\sum_0^T \delta^t w(\overline{C}_{t+1}) + \varepsilon \geqslant \sum_0^T \delta^t w(C_{t+1}^{n'}) + 3(\varepsilon/5). \tag{*}$$

Since $\{K_t, A_t, h_{t+1}\}_0^\infty$ is the optimal plan from $(\mathbf{K}, \mathbf{A})$, given the belief $\{E_t\}_0^\infty$, we have $C_{t+1} > 0$ for $t \geqslant 0$ (by using the fact that $w'(C) \to \infty$ as $C \to 0$). Then, there is $N_2 \geqslant N_1$ such that, for all $n' \geqslant N_2$, the sequence $\{\tilde{K}_t^{n'}, \tilde{A}_t^{n'}, \tilde{h}_{t+1}^{n'}\}_0^\infty$, defined by $(\tilde{K}_t^{n'}, \tilde{A}_t^{n'}, \tilde{h}_{t+1}^{n'}) = (K_t, A_t, h_{t+1})$ for $0 \leqslant t \leqslant T-1$, $(\tilde{K}_t^{n'}, \tilde{A}_t^{n'}, \tilde{h}_{t+1}^{n'}) = (K_T, A_T, 1)$ for

$t \geqslant T$, is a plan from $(\mathbf{K}, \mathbf{A})$, given the belief $\{E_t^{n'}\}_0^\infty$. Note that, for $0 \leqslant t \leqslant T-1$, $\tilde{C}_{t+1}^{n'} \to C_{t+1}$ as $n' \to \infty$. Thus, we can find $N \geqslant N_2$ such that, for $n' \geqslant N$, we have

$$\sum_0^{T-1} \delta^t w(\tilde{C}_{t+1}^{n'}) - \sum_0^{T-1} \delta^t w(C_{t+1}) \geqslant -(\varepsilon/5).$$

This in turn implies that, for $n' \geqslant N$,

$$\sum_0^\infty \delta^t w(\tilde{C}_{t+1}^{n'}) \geqslant \sum_0^\infty \delta^t w(C_{t+1}) - 2(\varepsilon/5). \qquad (**)$$

Combining $(+)$, $(*)$ and $(**)$ yields for $n' \geqslant N$

$$\sum_0^\infty \delta^t w(\tilde{C}_{t+1}^{n'}) \geqslant \sum_0^\infty \delta^t w(C_{t+1}^{n'}) + (\varepsilon/5),$$

which contradicts the fact that $\{K_t^{n'}, A_t^{n'}, h_{t+1}^{n'}\}_0^\infty$ is the optimal plan from $(\mathbf{K}, \mathbf{A})$, given the belief $\{E_t^{n'}\}_0^\infty$. Thus, continuity of ψ is ensured.

Let π be the (projection) map that associates with each plan $\{K'_t, A'_t, h'_{t+1}\}_0^\infty$ from $(\mathbf{K}, \mathbf{A})$, given the belief $\{E'_t\}_0^\infty$ in F, the sequence $\{A'_t\}_0^\infty$. Then π is a continuous map. Let Λ be the composite map $\pi \circ \psi$. Then Λ is continuous on F. Thus, applying the Tychonoff fixed-point theorem (see Smart 1974, p. 15), there is $\{\hat{E}_t\}_0^\infty$ in F, such that $\Lambda(\{\hat{E}_t\}_0^\infty) = \{\hat{E}_t\}_0^\infty$. Let $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty = \psi(\{\hat{E}_t\}_0^\infty)$. Then $\{\hat{A}_t\}_0^\infty = \pi \circ \psi(\{\hat{E}_t\}_0^\infty) = \Lambda(\{\hat{E}_t\}_0^\infty) = \{\hat{E}_t\}_0^\infty$. Thus, $\{\hat{K}_t, \hat{A}_t, \hat{h}_{t+1}\}_0^\infty$ is an equilibrium programme from $(\mathbf{K}, \mathbf{A})$. ■

Proof of Proposition 2

This is by now a fairly straightforward exercise. A detailed argument can be written down following Mitra and Sihag (1993). ■

Proof of Proposition 3

If the proposition were not true, there would exist an equilibrium $\{K_t, A_t, h_{t+1}\}$ from some $(\mathbf{K}, \mathbf{A}) \gg 0$ and $T \geqslant 0$, such that $A_{t+1} = A_t = A$ for $t \geqslant T$. In this case, one can use the δ-productivity condition (D) and apply the arguments used in Mitra and Sihag (1993) to conclude that the agent could improve his discounted sum of welfares by increasing his human capital to $(A + \varepsilon)$ for $t > T$, for $\varepsilon > 0$ but sufficiently small. ■

Proof of Proposition 4

As in the case of Proposition 2, this is a straightforward exercise. The argument is analogous to that spelt out in detail in Mitra and Sihag (1993). ■

Proof of Proposition 5

If $\{K_t, A_t, h_{t+1}\}_0^\infty$ is an interior equilibrium from $(\mathbf{K}, \mathbf{A}) \gg 0$, then we can use Propositions 2 and 4 to obtain, for $t \geqslant 1$,

$$(1/\delta)(C_{t+1}/C_t)^\sigma = (1 + \beta A_t^\gamma K_t^{\beta-1} h_{t+1}^{1-\beta} A_t^{1-\beta}), \tag{A1}$$

$$(1/\delta)(C_{t+1}/C_t)^\sigma = \frac{A_t^\gamma K_t^\beta (1/v)^{1-\beta}(1-\beta)[(1+v)A_t - A_{t+1}]^{-\beta}(1+v)}{A_{t-1}^\gamma K_{t-1}^\beta (1/v)^{1-\beta}(1-\beta)[(1+v)A_{t-1} - A_t]^{-\beta}}. \tag{A2}$$

Noting that $A_t^\gamma K_t^{\beta-1} h_{t+1}^{1-\beta} A_t^{1-\beta} = A_t^{\mu(1-\beta)} h_{t+1}^{1-\beta}/K_t^{1-\beta}$, and using the notation $x_t = A_t^\mu$, (A1) can be simplified to read

$$(1/\delta)(C_{t+1}/C_t)^\sigma = [1 + \beta(h_{t+1}x_t/K_t)^{1-\beta}].$$

Note that

$$A_t^\gamma K_t^\beta [(1+v)A_t - A_{t+1}]^{-\beta} = A_t^\gamma (vA_t)^{-\beta} K_t^\beta \{[(1+v)A_t - A_{t+1}]/vA_t\}^{-\beta}$$

$$= v^{-\beta} K_t^\beta / h_{t+1}^\beta A_t^{\beta-\gamma}.$$

Furthermore,

$$\mu\beta - [\gamma/(1-\beta)] = [\beta(1-\beta+\gamma) - \gamma]/(1-\beta)$$

$$= [\beta(1-\beta) - \gamma(1-\beta)]/(1-\beta) = (\beta - \gamma).$$

So we can write $A_t^{\beta-\gamma} = A_t^{\mu\beta}/A_t^{\gamma/(1-\beta)}$. Using this in (A2), we obtain

$$(1/\delta)(C_{t+1}/C_t)^\sigma = (1+v)\frac{(h_t x_{t-1}/K_{t-1})^\beta}{(h_{t+1}x_t/K_t)^\beta}(A_t/A_{t-1})^{\gamma/(1-\beta)}.$$

Noting that $\mu - 1 = [(1-\beta+\gamma)/(1-\beta)] - 1 = \gamma/(1-\beta)$, we obtain (ii) of Proposition 5. ■

Proof of Theorem 2

We first note that $\{K_t, A_t, h_{t+1}\}_0^\infty$ as defined is a plan from $(\mathbf{K}, \mathbf{A})$ given the beliefs $\{A_t\}_0^\infty$.

Note that assumption (C) ensures that $g < (1+v)$ (and $G < (1+v)^\mu$), while assumption (D) ensures that $g > 1$ (and $G > 1$). Then, $h_{t+1} \equiv h \equiv 1 - [(g-1)/v]$ satisfies $0 < h < 1$. Thus, it remains to check that $K_{t+1} \leqslant F(K_t, A_t h_{t+1}, A_t) + K_t$ for $t \geqslant 0$. To verify this, we write

$$F(K_t, A_t h_{t+1}, A_t) + K_t = K_t[1 + (h_{t+1}A_t^\mu/K_t)^{1-\beta}]$$

$$= K_t[1 + (h\mathbf{A}^\mu/\mathbf{K})^{1-\beta}] > K_t[1 + \beta\{[(1/\delta)G^\sigma - 1]/\beta\}]$$

$$= K_t[G^\sigma/\delta] = GK_t[G^\sigma/\delta G] = GK_t/\delta G^{1-\sigma} > GK_t = K_{t+1},$$

by using assumption (C).

Next, it is easy to verify that $\{K_t, A_t, h_{t+1}\}_0^\infty$ is a steady-state plan from $(\mathbf{K}, \mathbf{A})$ given the beliefs $\{A_t\}_0^\infty$. By definition $A_t = \mathbf{A}g^t$ and $K_t = \mathbf{K}G^t$. Also for $t \geqslant 1$, $C_{t+1} = A_t^\gamma h_{t+1}^{1-\beta} A_t^{1-\beta} K_t^\beta + K_t - K_{t+1} = K_t[1 + (h\mathbf{A}^\mu/\mathbf{K})^{1-\beta}] - GK_t = GK_{t-1}([1 + (h\mathbf{A}^\mu/\mathbf{K})^{1-\beta}] - G^2 K_{t-1} = GC_t$. Define $\mathbf{C} = \mathbf{K}[1 + (h\mathbf{A}^\mu/\mathbf{K})^{1-\beta}] - G\mathbf{K}$. Then $C_{t+1} = \mathbf{C}G^t$ for $t \geqslant 0$.

We now verify that $\{K_t, A_t, h_{t+1}\}_0^\infty$ satisfies the Ramsey–Euler equations. For $t \geqslant 1$, we have

$$(1/\delta)(C_{t+1}/C_t)^\sigma = G^\sigma/\delta$$

and

$$[1 + \beta(h_{t+1}A_t^\mu/K_t)^{1-\beta}] = G^\sigma/\delta,$$

so that the first Ramsey–Euler equation is satisfied. Also for $t \geqslant 1$,

$$\begin{aligned}(1+v)\frac{(h_t A_{t-1}^\mu/K_{t-1})^\beta}{(h_{t+1}A_t^\mu/K_t)^\beta}(A_t/A_{t-1})^{\mu-1}\\ &= (1+v)g^{\mu-1}\\ &= [\delta(1+v)]g^{\mu-1}/\delta = [\delta(1+v)]^{1+\{(\mu-1)/[1-\mu(1-\sigma)]\}}/\delta\\ &= [\delta(1+v)]^{\{\mu\sigma/[1-\mu(1-\sigma)]\}}/\delta = G^\sigma/\delta,\end{aligned}$$

so that the second Ramsey–Euler equation is satisfied.

Define for $t \geqslant 0$, $p_{t+1} = \delta^t w'(C_{t+1})$, $q_{t+1} = p_{t+1}[A_t^{\mu-1}(1-\beta)/v(h_{t+1}A_t^\mu/K_t)^\beta]$, $p_0 = p_1[1 + \beta(h_1\mathbf{A}^\mu/\mathbf{K})^{1-\beta}]$, $q_0 = q_1(1+v)$. Then, by the first Ramsey–Euler equation,

$$p_{t+1}[1 + \beta(h_{t+1}A_t^\mu/K_t)^{1-\beta}] = p_t \qquad \text{for } t \geqslant 0.$$

And, by the second Ramsey–Euler equation,

$$q_{t+1}(1+v) = q_t \qquad \text{for } t \geqslant 0.$$

Let $\{K'_t, A'_t, h'_{t+1}\}_0^\infty$ be any plan from $(\mathbf{K}, \mathbf{A})$ given the beliefs $\{A_t\}_0^\infty$. Then for $t \geqslant 0$, we have

$$\begin{aligned}\delta^t[w(C'_{t+1}) - w(C_{t+1})] &\leqslant \delta^t w'(C_{t+1})(C'_{t+1} - C_{t+1})\\ &= p_{t+1}(C'_{t+1} - C_{t+1})\\ &= p_{t+1}A_t^\gamma[K_t'^\beta(h'_{t+1}A'_t)^{1-\beta} - K_t^\beta(h_{t+1}A_t)^{1-\beta}]\\ &\quad + p_{t+1}(K'_t - K_t) - p_{t+1}(K'_{t+1} - K_{t+1})\\ &\leqslant p_{t+1}\left[1 + A_t^\gamma\beta\left(\frac{h_{t+1}A_t}{K_t}\right)^{1-\beta}\right](K'_t - K_t)\\ &\quad + p_{t+1}A_t^\gamma(1-\beta)\left(\frac{h_{t+1}A_t}{K_t}\right)^{-\beta}(h'_{t+1}A'_t - h_{t+1}A_t)\\ &\quad - p_{t+1}(K'_{t+1} - K_{t+1})\\ &= p_t(K'_t - K_t) - p_{t+1}(K'_{t+1} - K_{t+1})\\ &\quad + p_{t+1}[A_t^{\mu-1}(1-\beta)/v(h_{t+1}A_t^\mu/K_t)^\beta]\end{aligned}$$

$$\times [(1+v)(A'_t - A_t) - (A'_{t+1} - A_{t+1})]$$

$$= p_t(K'_t - K_t) - p_{t+1}(K'_{t+1} - K_{t+1})$$

$$+ q_t(A'_t - A_t) - q_{t+1}(A'_{t+1} - A_{t+1}).$$

Thus, summing on this inequality from $t = 0$ to $t = T$, we get

$$\sum_{t=0}^{T} \delta^t [w(C'_{t+1}) - w(C_{t+1})] \leqslant p_{T+1}(K_{T+1} - K'_{T+1}) + q_{T+1}(A_{T+1} - A'_{T+1})$$

$$\leqslant p_{T+1}K_{T+1} + q_{T+1}A_{T+1}.$$

Notice that $p_t = p_{t+1}(G^\sigma/\delta) = p_{t+1}G/\delta G^{1-\sigma}$. Thus, $p_{t+1}K_{t+1} = p_t K_t(\delta G^{1-\sigma})$, and $p_t K_t \to 0$ as $t \to \infty$ (since $\delta G^{1-\sigma} < 1$). Similarly, $q_t = q_{t+1}(1+v)$, and so $q_{t+1}A_{t+1} = q_t A_t[g/(1+v)]$. Thus, $q_t A_t \to 0$ as $t \to \infty$ (since $g < (1+v)$). This implies that

$$\sum_{0}^{\infty} \delta^t w(C'_{t+1}) \leqslant \sum_{0}^{\infty} \delta^t w(C_{t+1}),$$

and so $\{K_t, A_t, h_{t+1}\}_0^\infty$ is an optimal plan from (**K**, **A**), given the belief $\{A_t\}_0^\infty$. This establishes that $\{K_t, A_t, h_{t+1}\}_0^\infty$ is an equilibrium programme from (**K**, **A**). Hence, $\{K_t, A_t, h_{t+1}\}_0^\infty$ is an equilibrium steady state programme from (**K**, **A**). ■

Proof of Proposition 7

Using Proposition 5, and defining

$$z_t = (h_{t+1}x_t/K_t) \qquad \text{for } t \geqslant 0,$$

we obtain the equation

$$(1 + \beta z_t^{1-\beta}) = (z_{t-1}/z_t)^\beta (A_t/A_{t-1})^{\mu-1}(1+v).$$

This leads to the basic difference equation,

$$z_t^\beta + \beta z_t = z_{t-1}^\beta (A_t/A_{t-1})^{\mu-1}(1+v). \tag{A3}$$

In order to study (A3), we define a function $f(z) = M(z^\beta + \beta z)^{1/\beta}$ for $z \geqslant 0$, where $M > 0$. Then f is increasing and continuous in z, and the range of f is $\Re_+$. Thus, $\phi = f^{-1}$ exists and is an increasing continuous function from $\Re_+$ to $\Re_+$. We can infer a number of useful properties of ϕ from those of f. Since $f(0) = 0$, we have $\phi(0) = 0$. Also, f is continuously differentiable on $\Re_{++}$, with $f'(z) = \mathrm{M}(1/\beta)[\mathrm{z}^\beta + \beta \mathrm{z}]^{(1/\beta)-1}[\beta \mathrm{z}^{\beta-1} + \beta] = \mathrm{M}[1 + \beta \mathrm{z}^{1-\beta}]^{(1-\beta)/\beta}\ [1 + z^{1-\beta}] > 0$, so that ϕ is continuously differentiable on $\Re_{++}$ and $\phi'(y) = 1/f'(\phi(y)) > 0$ for all $y > 0$. Furthermore, as y increases, $\phi(y)$ increases and so $f'(\phi(y))$ increases and $\phi'(y)$ decreases. Thus, ϕ is strictly concave on $\Re_{++}$. As $y \to \infty$, $\phi(y) \to \infty$ and so $f'(\phi(y)) \to \infty$, and $\phi'(y) \to 0$. As $y \to 0$, $\phi(y) \to 0$ and so $f'(\phi(y)) \to M$ and $\phi'(y) \to (1/M)$.

We now study the dynamic behaviour of $\{z_t\}_0^\infty$ as governed by (A3), by looking at two related difference equations:

$$\xi_t^{\beta} + \beta \xi_t = \xi_{t-1}^{\beta}(1+v), \qquad \xi_0 = z_0, \tag{A4}$$

$$\zeta_t^{\beta} + \beta \zeta_t = \zeta_{t-1}^{\beta}(1+v)^{\mu}, \qquad \zeta_0 = z_0. \tag{A5}$$

Notice that (A4) translates to the equation $f(\xi_t) = \xi_{t-1}$ when $M = [1/(1+v)^{1/\beta}]$; similarly, (A5) translates to the equation $f(\zeta_t) = \zeta_{t-1}$ when $\underline{M} = [1/(1+v)^{\mu/\beta}]$. Let us denote f^{-1} in the first case by $\underline{\phi}$ and in the second case by $\overline{\phi}$.

Given that $1 \leqslant (A_t/A_{t-1}) \leqslant (\overline{1} + v)$ for all $t \geqslant 1$, we clearly have

$$\xi_t \leqslant z_t \leqslant \zeta_t \qquad \text{for } t \geqslant 0. \tag{A6}$$

Thus, we can infer the behaviour of z_t by examining those of ξ_t and ζ_t. The latter are governed by the equations (A4) and (A5), that is, by the equations

$$\xi_t = \underline{\phi}(\xi_{t-1}), \qquad \xi_0 = z_0, \tag{A7}$$

$$\zeta_t = \overline{\phi}(\zeta_{t-1}), \qquad \zeta_0 = z_0. \tag{A8}$$

Since $\underline{\phi}'(y)$ and $\overline{\phi}'(y)$ both approach numbers exceeding 1 ($M < 1$ in both cases) as $y \to 0$, the solutions of (A7) and (A8), given any $z_0 > 0$, are easy to describe. In each case, the solution approaches the unique non-zero fixed point of the function. Denoting by ξ^* and ζ^* these fixed points of $\underline{\phi}$ and $\overline{\phi}$ respectively, we have $\xi^* < \zeta^*$. Pick any $0 < \varepsilon < \xi^*$ and define $\underline{z} = \xi^* - \varepsilon$, $\overline{z} = \zeta^* + \varepsilon$, $\mathrm{I} = [\underline{z}, \overline{z}]$. Then, if $z_t \in I$ for some t, $z_{t+1} \in I$ also. Further, given any $z_0 > 0$, $z_t \in I$ for some $t \geqslant 0$. This proves the proposition. ■

Proof of Lemma 2

Suppose on the contrary that $(C_t/Y_t) \to 1$ along a subsequence of periods. Then $(K_t/Y_t) \to 0$ along that subsequence, and so $(C_t/K_t) \to \infty$ along the subsequence also.

By Proposition 7, for all t large, $z_t \in I$, and so by the first Ramsey–Euler equation

$$(C_{t+1}/C_t) = \delta^{1/\sigma}[1 + \beta z_t^{1-\beta}]^{1/\sigma}$$

is bounded below by a positive number. Thus, $(C_{t+1}/K_t) = (C_{t+1}/C_t)(C_t/K_t)$ must go to infinity along the subsequence also. However,

$$(C_{t+1}/K_t) = [1 + z_t^{1-\beta}] - (K_{t+1}/K_t) \leqslant [1 + z_t^{1-\beta}],$$

so that (C_{t+1}/K_t) is bounded above, a contradiction.

Proof of Proposition 8

We have $y_t = (x_t/K_t) = (z_t/h_{t+1}) \geqslant z_t$ for $t \geqslant 0$. By Proposition 7, $\liminf_{t\to\infty} z_t > 0$, and so $\liminf_{t\to\infty} y_t > 0$.

Since $y_t = (z_t/h_{t+1})$ for $t \geqslant 0$ and z_t is bounded above (by Proposition 7), if $\limsup_{t\to\infty} y_t = \infty$, then $\liminf_{t\to\infty} h_{t+1} = 0$.

We now show that a switch (sufficiently far in the future) to a regime of no schooling for the rest of the future is an improvement for this agent's discounted sum of welfares.

Let $\{E_t\}_0^{\infty}$ be any belief satisfying

$$E_t \leqslant E_{t+1} \leqslant (1+v)E_t \qquad \text{for } t \geqslant 0. \tag{A9}$$

Define, for $(K, A) \geqslant 0$ and any belief $\{E_t\}_0^\infty$ satisfying (A9),

$$W(K, A, \{E_t\}_0^\infty) = \text{Max} \sum_0^\infty \delta^t w(C_{t+1})$$

subject to

$$\text{for } t \geqslant 0 \begin{cases} C_{t+1} = F(K_t, A_t h_{t+1}, E_t) + K_t - K_{t+1} \\ A_{t+1} = A_t + v(1 - h_{t+1})A_t \\ 0 \leqslant h_{t+1} \leqslant 1, \ C_{t+1} \geqslant 0 \end{cases}$$

$$\text{and} \qquad (K_0, A_0) = (K, A).$$

Using the method employed to establish Proposition 1, W is well defined. Also, given $\{E_t\}_0^\infty$, it is easy to check (given the forms of the welfare and production functions) that W is homogeneous of degree $(1 - \sigma)$ in (K, A).

Now define a subsequence of time-periods as follows: $t_1 = 0$; for $s \geqslant 1$, $t_{s+1} = \min\{t: y_t > y_{t_s}\}$. Since $\limsup_{t\to\infty} y_t = \infty$, this is a well-defined subsequence. Also, for each $s \geqslant 1$, $y_{t_{s+1}} \geqslant y_t$ for all $0 \leqslant t \leqslant t_{s+1}$. Thus, $y_{t_s} \to \infty$ as $s \to \infty$. Furthermore, by construction $y_{t_s} \geqslant y_{t_s - 1}$ for all s. Thus, $(K_{t_s}/K_{t_s-1}) \leqslant (A_{t_s}/A_{t_s-1})^\mu \leqslant (1 + v)^\mu$ for all s.

Let T be sufficiently large so that, for $t \geqslant T$, $z_t \in I$. (This is assured by Proposition 7.) Now, for any $t > T$, $x_t = A_t^\mu \leqslant (1 + v)^\mu A_{t-1}^\mu = \theta x_{t-1}$.

In view of Lemma 2, there is $0 < \alpha < 1$, such that $(C_t/Y_t) \leqslant (1 - \alpha)$ for $t \geqslant 1$. Thus, $(K_t/K_{t-1}) = (Y_t - C_t)/K_{t-1} \geqslant [Y_t - (1 - \alpha)Y_t]/K_{t-1} = \alpha(Y_t/K_{t-1}) = \alpha[1 + z_{t-1}^{1-\beta}] \geqslant \alpha[1 + \underline{z}^{1-\beta}]$. Thus, $(x_{t-1}/K_{t-1}) \geqslant (x_t/\theta)[\alpha(1 + \underline{z}^{1-\beta})/K_t]$. Thus, $y_{t_s-1} \to \infty$ as $s \to \infty$, since $y_{t_s} \to \infty$ as $s \to \infty$. Furthermore, $h_{t_s} \to 0$ as $s \to \infty$ by Proposition 7.

Denote $\min[\underline{z}^{1-\beta}/2, 1/2]$ by m. Choose s large enough so that

$$(m/h_{t_s}^{1-\beta})^{1-\sigma} > 1/(1 - \delta\theta^{1-\sigma}). \tag{A10}$$

Using the definition of W, we obtain

$$\begin{aligned} W[K_{t_s-1}, A_{t_s-1}, \{A_{t_s-1+n}\}_0^\infty] &= w(C_{t_s}) + \delta W[K_{t_s}, A_{t_s}, \{A_{t_s+n}\}_0^\infty] \\ &\leqslant w(C_{t_s}) + \delta W[K_{t_s}, A_{t_s}, \{(1+v)A_{t_s-1+n}\}_0^\infty] \\ &\leqslant w(C_{t_s}) + \delta W[K_{t_s}, (1+v)^{\mu-1}A_{t_s}, \{A_{t_s-1+n}\}_0^\infty] \\ &\leqslant w(C_{t_s}) + \delta W[(1+v)^\mu K_{t_s-1}, (1+v)^\mu A_{t_s-1}, \{A_{t_s-1+n}\}_0^\infty] \\ &= w(C_{t_s}) + \delta[(1+v)^\mu]^{1-\sigma} W[K_{t_s-1}, A_{t_s-1}, (A_{t_s-1+n})_0^\infty]. \end{aligned}$$

Thus, $W[K_{t_s-1}, A_{t_s-1}, \{A_{t_s-1+n}\}_0^\infty] \leqslant w(C_{t_s})/(1 - \delta\theta^{1-\sigma})$.

We now show that, by using all the labour in period t_s in the production of physical goods and by consuming all of the output, a higher discounted sum of welfares can be obtained. Denote period t_s by τ, and the new consumption by C'_τ. Then

$$C'_\tau \geqslant F(K_{\tau-1}, A_{\tau-1}h_\tau, A_{\tau-1})/h_\tau^{1-\beta}$$

$$\geqslant \frac{z_{\tau-1}^{1-\beta}K_{\tau-1}}{2h_\tau^{1-\beta}} + \frac{F(K_{\tau-1}, A_{\tau-1}h_\tau, A_{\tau-1})}{2h_\tau^{1-\beta}}$$

$$\geqslant (m/h_\tau^{1-\beta})[K_{\tau-1} + F(K_{\tau-1}, A_{\tau-1}h_\tau, A_{\tau-1})]$$

$$\geqslant (mC_\tau/h_\tau^{1-\beta}).$$

Thus, $w(C'_\tau) \geqslant (m/h_\tau^{1-\beta})^{1-\sigma} C_\tau^{1-\sigma}/(1-\sigma) > w(C_\tau)/(1-\delta\theta^{1-\sigma})$, by using (A10). This contradicts the definition of W, and establishes the result. ■

Proof of Lemma 3

Using condition (C), we have $\delta[(1+v)^\mu]^{1-\sigma} < 1$. Thus, we can find $\alpha > 1$, such that $\lambda' \equiv \delta[(1+v)^\mu]^{1-\sigma}\ \alpha < 1$. Denote $[\lambda']^{1/\sigma}$ by λ. Using Proposition 7, we can find T such that, for $t \geqslant T$, $z_t \in I$ and, further, $(1+\beta z_t^{1-\beta}) \leqslant \alpha(1+v)^\mu$. Thus, using the first Ramsey–Euler equation for $t \geqslant T$,

$$(C_{t+1}/C_t) = \delta^{1/\sigma}(1+\beta z_t^{1-\beta})^{(1/\sigma)-1}(1+\beta z_t^{1-\beta})$$

$$\leqslant \delta^{1/\sigma}\alpha^{1/\sigma}[(1+v)^\mu]^{(1/\sigma)-1}[1+\beta z_t^{1-\beta}]$$

$$= \{\delta[(1+v)^\mu]^{1-\sigma}\alpha\}^{1/\sigma}(1+\beta z_t^{1-\beta})$$

$$= \lambda(1+\beta z_t^{1-\beta}) < \alpha(1+v)^\mu.$$

Now, if $(C_t/K_t) \to 0$ along a subsequence of periods, we can pick $\tau > T$ along this subsequence so that $(C_\tau/K_\tau) \leqslant (1-\beta)z^{1-\beta}/\alpha(1+v)^\mu$. Then $(C_{\tau+1}/K_\tau) \leqslant (C_{\tau+1}/C_\tau)(C_\tau/K_\tau) \leqslant (1-\beta)z^{1-\beta}$.

Now, $(K_{\tau+1}/K_\tau) \geqslant (1+z_\tau^{1-\beta}) - (C_{\tau+1}/K_\tau) \geqslant (1+\beta z_\tau^{1-\beta}) + (1-\beta)z_\tau^{1-\beta} - (C_{\tau+1}/K_\tau) \geqslant (1+\beta z_\tau^{1-\beta}) + (1-\beta)\underline{z}^{1-\beta} - (C_{\tau+1}/K_\tau) \geqslant (1+\beta z_\tau^{1-\beta})$. Thus, $(C_{\tau+1}/K_{\tau+1}) = (C_{\tau+1}/C_\tau)(C_\tau/K_\tau)/(K_{\tau+1}/K_\tau) \leqslant \lambda(C_\tau/K_\tau)$. Thus, this step can be repeated to get

$$(C_{\tau+s}/K_{\tau+s}) \leqslant \lambda^s(C_\tau/K_\tau) \qquad \text{for } s \geqslant 1.$$

This implies $\sum_{t=1}^{\infty}(C_t/K_t) < \infty$. ■

Proof of Proposition 9

We have $v_t = (C_{t+1}/K_t) = (C_{t+1}/Y_{t+1})(Y_{t+1}/K_t) \leqslant (1+z_t^{1-\beta})$, which is bounded above by Proposition 7. Thus,

$$\limsup_{t\to\infty} v_t < \infty$$

To establish that $\liminf_{t\to\infty} v_t > 0$, suppose on the contrary that $(C_{t+1}/K_t) \to 0$ along a subsequence of periods. We can use Proposition 7 to find T such that, for $t \geqslant T$, $z_t \in I$. And for $t \geqslant T$, using the first Ramsey–Euler equation,

$$(C_{t+1}/C_t) = \delta^{1/\sigma}(1+\beta z_t^{1-\beta})^{1/\sigma} \geqslant \delta^{1/\sigma}(1+\beta\underline{z}^{1-\beta})^{1/\sigma}.$$

Thus $(C_t/K_t) = (C_{t+1}/K_t)/(C_{t+1}/C_t) \to 0$ along the given subsequence of periods. Thus, by Lemma 3,

$$\sum_{t=1}^{\infty}(C_t/K_t) < \infty.$$

Then, following the proof of lemma 2 in Mitra (1979, p. 89), we can find a plan $\{K'_t, A_t, h_{t+1}\}_0^\infty$ from (**K**, **A**), given the belief $\{A_t\}_0^\infty$ such that $C'_t \geqslant C_t$ for all $t \geqslant 1$ and $C'_t > C_t$ for some t. This contradicts the fact that $\{K_t, A_t, h_{t+1}\}_0^\infty$ is an equilibrium programme from (**K**, **A**).

Proof of Theorem 3

Define $g_K = \liminf_{t\to\infty}(K_t)^{1/t}$, $g_A = \liminf_{t\to\infty}(A_t)^{1/t}$ and $g_C = \liminf_{t\to\infty}(C_t)^{1/t}$. Then, using Proposition 8, we get $g_K = g_A^\mu$; and by using Proposition 9, $g_C = g_K$. Using the second Ramsey–Euler equation, we obtain

$$(C_{T+1}/C_1)^\sigma = [\delta(1+v)]^T (z_0/z_T)^\beta (A_T/A_0)^{\mu-1}.$$

Then, taking lim inf on both sides of the equation (and using Proposition 7),

$$g_C^\sigma = \delta(1+v) g_A^{\mu-1}.$$

Using $g_K = g_C = g_A^\mu$, we have

$$g_A^{\mu\sigma} = \delta(1+v) g_A^{\mu-1}$$

Thus, $g_A = [\delta(1+v)]^{1/[1-\mu(1-\sigma)]} = g$, the growth factor of human capital for the equilibrium steady state. And $g_K = g_C = g_A^\mu = g^\mu = G$, the growth factor of physical capital (and consumption) for the equilibrium steady state.

Repeating the above steps by defining $g_K = \limsup_{t\to\infty}(K_t)^{1/t}$, $g_A = \limsup_{t\to\infty}(A_t)^{1/t}$, $g_C = \limsup_{t\to\infty}(C_t)^{1/t}$, we again get $g_A = g$ and $g_K = g_C = G$. Thus, $\lim_{t\to\infty}(A_t)^{1/t}$ exists and equals g; similarly, $\lim_{t\to\infty}(K_t)^{1/t}$ and $\lim_{t\to\infty}(C_t)^{1/t}$ exist and equal G.

Final version accepted October 28, 1996.

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